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ON THE WEAK OBSERVABILITY OF SMALL SOLUTIONS OF DIFFERENTIAL-ALGEBRAIC SYSTEMS WITH DELAYS[1]

The paper considers the problem of observability of small solutions for hybrid time invariant differential-difference dynamic systems, i. e. linear stationary differential-algebraic systems with delays (DAD systems). Several types of observability of small solutions are defined and the corresponding parametric criteria are given. Spectral observability is considered and relation of the spectral observability to the observability of small solutions is discussed.

Introduction. The behaviour of a number of real physics process consists of a combination of dynamic (differential) and algebraic (functional) dependencies. These processes are described by differential-algebraic (DAE) systems. In that sense these systems are hybrid systems. It should be noted that the term «hybrid systems» has been widely used in the literature in various senses [1].

The paper deals with the weak observability of small solutions of DAD systems; it is an extension of the work [2]. The small solution is a solution that goes to zero faster than any exponential function. Existence of such solutions for linear retarded systems was proved by Henry [3] and later by Kappel [4] for linear neutral type systems. Lunel [5] gave explicit characterization of the smallest possible time for which small solutions vanish. Observability of small solutions for the retarded time delay system case was first studied by Manitius [6] and for general neutral system by Salomon [7].

1. Preliminaries. Let us consider DAD system in the form

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t),\ t > 0, \quad (1)$$

$$x_2(t) = A_{21}x_1(t) + A_{22}x_2(t-h),\ t \ge 0, \quad (2)$$

with output

$$y(t) = B_1x_1(t) + B_2x_2(t), \quad (3)$$

Here $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $y(t) \in \mathbb{R}^m$, $t \ge 0$; $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{r\times n_1}$, $B_2 \in \mathbb{R}^{r\times n_2}$ are constant matrices h is a constant delay, $h > 0$. We regard an absolute continuous n_1-vector functions $x_1(\cdot)$, and a piecewise continuous $x_2(\cdot)$ n_2-vector functions as the solutions of systems (1)–(3), if they satisfy the equation (1) for almost all $t > 0$ and (2) for all $t \ge 0$. System (1)–(3) should be completed with initial conditions in the form

$$x_1(0+) = x_1(0) = x_{10},\quad x_2(\tau) = \psi(\tau), \tau \in [-h, 0), \quad (4)$$

where $\psi \in PC([-h,0), \mathbb{R}^m)$ and $PC([-h,0), \mathbb{R}^m)$ is a set of piecewise continuous m-vector functions in $[-h, 0]$.

Let $E(g)$ denote the exponential type of $g : \mathbb{C} \to \mathbb{C}$, assuming g is an entire function of order 1. Then

$$E(g) = \limsup_{|s|\to\infty} \frac{\log|g(s)|}{|s|}.$$

For $g : \mathbb{C} \to \mathbb{C}^q$ the exponential type of g is defined by

$$E(g) = \max_{1\le j\le q} E(g_j), \text{ where } g = \begin{bmatrix} g_1 & \dots & g_q \end{bmatrix}^T.$$

Let $\Delta(p)$ be the characteristic matrix function

$$\Delta(p) = \begin{pmatrix} pI_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22}e^{-ph} \end{pmatrix}.$$

The matrix function $\Delta(p)$ appears by applying the Laplace transform to system (1)–(3). Let $\det\Delta(p)$ be the determinant of $\Delta(p)$. It follows from the above that the exponential type of $\det\Delta(p)$ is less or equal n_2h. Define ε by

$$E(\det\Delta(p)) = n_2h - \varepsilon.$$

Let $\operatorname{adj}\Delta(p)$ be the matrix function of cofactors of $\Delta(p)$. Since the cofactors C_{ij} are $(n_1+n_2-1)(n_1+n_2-1)$ subdeterminants of $\Delta(p)$, the exponential type of the cofactors is less or equal n_2h. Define σ by

$$\max_{1\le i,j\le n_1+n_2} E(C_{ij}) = n_2h - \sigma.$$

We have [2].

Proposition 1. For $x_1(\cdot)$, $x_2(\cdot)$ being solutions of system (1)–(3) the following implications hold:

i) if $\forall k \in \mathbb{Z}$ $x_1(t)e^{kt} \to 0$ as $t \to +\infty$, then $x_1(t) = 0$ for all $t \ge \varepsilon - \sigma$; (5)

ii) if $\forall k \in \mathbb{Z}$ $x_2(t)e^{kt} \to 0$ as $t \to +\infty$, then $x_2(t) = 0$ for all $t \ge \varepsilon - \sigma$. (6)

Definition 1. We say that a solution $x_1(\cdot)$, $x_2(\cdot)$ is small, if there exists $T > 0$ such that $x_1(t) = 0$, $x_2(t) = 0$ for $t \ge T$. A small solution is trivial, if it is zero for $t > 0$.

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Definition 2. We say that system (1)–(3) has a nontrivial small solutions, if there exists a solution $x_1(\cdot),\ x_2(\cdot)$ such that conditions (5), (6) hold and at least $x_1(\cdot)$ or $x_2(\cdot)$ is not trivial.

Definition 3. We say that system (1)–(3) has a nontrivial small solution with respect to x_1, if there exists a solution $x_1(\cdot),\ x_2(\cdot)$ such that condition (5) holds and $x_1(\cdot)$ is not trivial.

Definition 4. We say that system (1)–(3) has a nontrivial small solution with respect to x_2, if there exists a solution $x_1(\cdot),\ x_2(\cdot)$ such that condition (6) holds and $x_2(\cdot)$ is not trivial.

2. Observability.

2.1. Observability of small solutions.

Definition 5. We say that nontrivial small solutions of system (1)–(3) are observable, if every nontrivial small solution has nonzero output for some $t > 0$. This means that

$$\exists T>0 \left.\begin{array}{c} x_1(t)=0\,\forall t\ge T \\ x_2(t)=0\,\forall t\ge T \\ y(t)=0\,\forall t>0 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x_1(t)=0, x_2(t)=0, \\ \forall t>0. \end{array}\right.$$

Theorem 1. Nontrivial small solutions of system (1)–(3) are observable, if and only if the following conditions hold:

$$\text{i) } \max_{\lambda\in\mathbb{C}} \operatorname{rank}\begin{bmatrix} A_{11}-\lambda I_{n_1} & A_{12} & 0 \\ A_{21} & -I_{n_2} & A_{22} \\ 0 & A_{22} & 0 \\ B_1 & B_2 & 0 \end{bmatrix} = n_1+n_2+\operatorname{rank}A_{22}, \quad (7)$$

$$\text{ii) } \operatorname{rank}\begin{bmatrix} B_2A_{22} \\ B_2\left(A_{22}\right)^2 \\ \vdots \\ B_2\left(A_{22}\right)^{n_2} \\ \left(A_{22}\right)^{n_2} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2A_{22} \\ B_2\left(A_{22}\right)^2 \\ \vdots \\ B_2\left(A_{22}\right)^{n_2} \\ \left(A_{22}\right)^{n_2} \\ A_{22} \end{bmatrix}. \quad (8)$$

Proof. The proof is similar to theorem 2 and it can be omitted.

2.2. Spectral observability.

Definition 6. System (1)–(3) is infinite-time observable, if for all initial data for which $y(t)=0$ for $t\in[0,\infty)$ there exists t_1 such that $x_1(t)=0$ and $x_2(t)=0$ for $t\in[t_1, \infty)$.

Definition 7. System (1)–(3) is finite-time observable at t_2, if for all initial data, for which $y(t)=0$ for $t\in[0,\infty)$, $x_1(t)=0$ and $x_2(t)=0$ for $t\in[t_2,\infty)$.

Definition 8. System (1)–(3) is spectrally observable, if all its eigenvalues are observable. An eigenvalue λ is observable if the corresponding eigensolution of the form $x_1(t)=\exp(\lambda t)x_1(0)$, $x_2(t)=\exp(\lambda t)x_2(0)$, $x_1(0)\neq 0$, $x_2(0)\neq 0$, obtains $y(t)=0$ for $t\in[0, \infty)$.

We have [2].

Proposition 2. System (1)–(3) is spectrally observable if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda I_{n_1}-A_{11} & -A_{12} \\ -A_{21} & I_{n_2}-A_{22}e^{-\lambda h} \\ B_1 & B_2 \end{pmatrix} = n_1+n_2, \quad (9)$$

for all complex λ.

Proposition 3. System (1)–(3) is spectrally observable if and only if system (1)–(3) is infinite-time observable.

Corollary 1. System (1)–(3) is spectrally observable if and only if system (1)–(3) is finite-time observable at $\varepsilon-\sigma$.

Proof. By proposition 1 and proposition 3.

3. Relative observability of small solutions.

Definition 9. Nontrivial small solutions with respect to x_2 of system (1)–(3) are weakly observable, if every nontrivial small solution with respect to x_2 has nonzero output for $t>0$ and x_1 is a zero solution, i. e.

$$\exists T>0 \left.\begin{array}{c} x_1(t)=0\,\forall t>0 \\ x_2(t)=0\,\forall t\ge T \\ y(t)=0\,\forall t\ge 0 \end{array}\right\} \Rightarrow x_2(t)=0,\ \forall t>0$$

Theorem 2. Nontrivial small solutions with respect to x_2 of system (1)–(3) are observable if and only if the following condition holds:

$$\operatorname{rank}\begin{bmatrix} B_2A_{22} \\ B_2(A_{22})^2 \\ \vdots \\ B_2(A_{22})^{n_2} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} B_2A_{22} \\ B_2(A_{22})^2 \\ \vdots \\ B_2(A_{22})^{n_2} \\ A_{22} \end{bmatrix}. \quad (10)$$

Proof. *The necessary condition.* We assume that $x_1(t)\equiv 0,\ t>0,\ \varphi(\tau)=\begin{cases} 0,\ \tau\in(-h,0), \\ \varphi_0,\ \tau=-h. \end{cases}$ Then equation (1) is satisfied for almost all $t>0$ and weak observability with respect to x_2 of system (1)–(3) means that conditions $B_2A_{22}\varphi_0=0,\ \ldots,\ B_2(A_{22})^k\varphi_0=0$ for $k=1,2,\ldots$ implies $A_{22}\varphi_0=0$ that by the Cayley – Hamilton theorem is equivalent to condition (10).

The sufficient condition. If condition (10) is satisfied then there exists a matrix $D\in\mathbb{R}^{n_2\times rn_2}$ such that $A_{22}=D\begin{bmatrix} B_2A_{22} \\ B_2(A_{22})^2 \\ \vdots \\ B_2(A_{22})^{n_2} \end{bmatrix}$. For any initial function $\varphi(\tau), \tau \in [-h, 0]$ for which $B_2(A_{22})^k \varphi(\tau) \equiv 0$, $\tau \in [-h, 0]$, $k = 1, \ldots, n_2$ condition $A_{22}\varphi(\tau) \equiv 0$, $\tau \in [-h, 0]$ is also satisfied that is equivalent to the weak observability of nontrivial small solutions of system (1)–(3) with respect to x_2.

Definition 10. We say that $x_1(t),\ t > 0$, $x_2(t),\ t > 0$ is a strong solution of system (1)–(3), if equations (1)–(3) are satisfied for all $t,\ t \geq 0$ (the derivative in (1) we means right-hand derivative at $t = 0$).

Theorem 3. Nontrivial strong small solutions with respect to x_2 for system (1)–(3), are observable if and only if the following condition holds:

$$\operatorname{rank}\begin{bmatrix} A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n_2} \\ B_2A_{22} \\ \vdots \\ B_2A_{22}^{n_2} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{12}A_{22} \\ \vdots \\ A_{12}A_{22}^{n_2} \\ B_2A_{22} \\ \vdots \\ B_2A_{22}^{n_2} \\ A_{22} \end{bmatrix}. \qquad (11)$$

Proof. Definition 10 is equivalent to

$$\left[\left.\begin{array}{c} A_{12}x_1(t) = 0,\ t > 0 \\ x_2(t) = A_{22}x_2(t-h),\ t > 0 \\ y(t) = B_2x_2(t),\ t > 0 \end{array}\right\} \Rightarrow x_2(t) = 0,\ t > 0 \right] \Leftrightarrow$$

$$\Leftrightarrow \left[\left.\begin{array}{c} A_{12}(A_{22})^i \varphi(\tau) = 0 \\ B_2(A_{22})^j \varphi(\tau) = 0 \end{array}\right\} \Rightarrow A_{22}\varphi(\tau) = 0,\right.$$

where

$$\tau \in [-h, 0),\ i = 1, \ldots, n_2;\ j = 1, \ldots, n_2 \big] \Leftrightarrow$$

$$\Leftrightarrow \{\varphi^T(\tau)[(A_{22}^T)^i A_{12}^T, (A_{22}^T)^j B_2^T,$$

$$i = 1, \ldots, n_2;\ j = 1, \ldots, n_2] \Rightarrow \varphi^T(\tau)A_{22}^T = 0\}.$$

It is equivalent to (11).

Corollary 2. If nontrivial strong small solutions with respect to x_2 of system (1)–(3) are observable then nontrivial small solutions with respect to x_2 of the system are also observable.

Definition 9. Nontrivial small solutions with respect to x_1 of system (1)–(3) are weakly observable if every non trivial small solution with respect to x_1 has nonzero output for $t > 0$ and for x_2 being zero solution, i. e.

$$\left.\begin{array}{r} x_2(t) = 0\ \forall t > 0 \\ \exists T > 0\ \ x_1(t) = 0\ \forall t \geq T \\ y(t) = 0\ \forall t > 0 \end{array}\right\} \Rightarrow x_1(t) = 0, \forall t > 0.$$

Theorem 4. Nontrivial small solutions with respect to x_1 of system (1)–(3) are always observable.

Proof. Condition $x_2(t) = 0$ for $t > 0$ implies $\dot{x}_1(t) = A_{11}x_1(t)$, and the system has solutions of the form $x_1(t) = e^{A_{11}t}x_1(0)$, it proves that all small solutions with respect to x_1 are trivial.

Conclusion. In this paper we investigated the problem of relative weak observability of nontrivial small solutions of the hybrid differential-difference (HDR) systems. Weak observability of nontrivial small solution with respect to x_2 and x_1 are considered. Strong small solutions are defined and weak observability of nontrivial strong small solutions with respect to x_2 is established. Other types of observability and relations between these types of observability are discussed.

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