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LAYERED SYSTEM DEFORMATION DETERMINATION: MATERIALS RHEOLOGICAL PROPERTIES CONSIDERATION

The author proposes a solution for the problem of pavement and roadbed deformation determination considering the rheological properties of materials, based on the mathematical elasticity theory and creep theory problems solving, using the mathematical models of the processes to be investigated. The time coordinate integral transformation was applied to solve the problem.

Introduction. In several publications (V.F. Babkov, A.K. Birulya, N.N. Ivanov, B.I. Kogan, M.B. Korsunsky etc.), various solutions have been proposed for the layered pavement design. However, the material creep is not adequately considered for this purpose. In 1961, M.B. Korsunsky [1] has formulated the ways to consider the materials creep for pavement design problems. Assuming that the modulus of elasticity behavior as a function of the loading rate and the load time is known, he managed to reduce the creep theory problem to the known elasticity theory problems. Later, B.S. Radovsky [2] has formulated the stress strain state problem for a multilayer viscoelastic half-space and has resolved it in an integral form for a uniform viscoelastic half-space. In 1969, I.A. Mednikov [3] has taken the materials creep into consideration to solve the beam bending problem for an infinitely long viscoelastic beam on a viscoelastic foundation, and the author has noted that, under several assumptions, this problem can be reduced to the pavement design problem. Thus, it's quite important to investigate the stress and movement variations in layered systems, taking into consideration the rheological properties of materials used in road construction.

Main part. The analysis has demonstrated that the technical literature does not adequately deal with the investigations of the stress strain state for layered systems, with the material creep taken into consideration and with the mathematically rigorous problem formulation.

The experiments were carried out to plot the creep curves, the analog computer was used to solve the differential equations, and the solutions were compared to choose and validate the deformation law, taking the time coordinate into consideration, for the most common road construction materials.

For the materials used in road construction, the equation as follows was found to be sufficiently precise for practical purposes:

$$En\frac{d\varepsilon}{dt} + H\varepsilon = n\frac{d\sigma}{dt} + \sigma, \qquad (1)$$

where E is the instant module of elasticity; n is the relaxation time factor depending on the material's viscoelastic properties and the loading mode; ε is the strain; H is the long-term modulus of elasticity; σ is the stress.

The multilayer quasi-static viscoelastic half-space, with the load on its surface uniformly distributed within the circle area, is applicable as a mathematical design model of pavements and roadbeds. Each (ith) layer is described by the parameters as follows: E_i, H_i, n_i, Poisson ratio (μ_i) and thickness (h_i).

Let's denote the movement tensor components in cylindrical coordinates by w and u, the movement along z and r axis respectively.

To solve the problem, the set of biharmonic functions shall be found, $\phi_i(r,z)$, with the equations describing the movement tensor's horizontal and vertical deformations in terms of these functions are as follows:

$$w_i = \frac{1+\mu_i}{E_i}\{[2(1-\mu_i)\nabla^2\varphi_i - \frac{\partial^2\varphi_i}{\partial z^2}] + \frac{E_i - H_i}{E_i n_i} \times$$
$$\times\int_0^t [\,2(1-\mu_i)\nabla^2\varphi_i - \frac{\partial^2\varphi_i}{\partial z^2}\,]\, e^{-\frac{H_i(t-\tau)}{E_i n_i}}\, d\tau\}, \qquad (2)$$
$$u_i = -\frac{1-\mu_i}{E_i}[\frac{\partial^2\varphi_i}{\partial z\partial r} + \frac{E_i - H_i}{E_i n_i}\int_0^t \frac{\partial^2\varphi_i}{\partial z\partial r} e^{-\frac{H_i(t-\tau)}{E_i n_i}}\, d\tau].$$

As is known, the desired function, $\phi_i(r,z)$, is acceptable if it meets the equation as follows:

$$\nabla^2\nabla^2\varphi_i(r,z) = 0, \qquad (3)$$

where ∇^2 – is the symbol similar to the Laplace operator and the boundary conditions.

Within each layer, the desired function $\phi_i(r,z)$ is continuous; for each ith layer, it can be expressed analytically as follows:

$$\varphi_i(r,z) = \int_0^\infty \{A + B[\alpha(\eta-1) + 2\mu_i] + \sum_{k=2}[C_k[(1-$$
$$-2\mu_i)(1-e^{-2\lambda_k}) + \lambda_k(1+e^{-2\lambda_k})] + D_k[2\mu_i(1+ \qquad (4)$$
$$+e^{-2\lambda_k}) - \lambda_k(1-e^{-2\lambda_k})]\}\}e^{-\alpha\eta}J_0(\rho\alpha)\,d\alpha,$$

where $\eta = z / h$; $\rho = r / h$; $\lambda_k = (h_{k-1} - z) / h$; h_i is the total thickness of layers above the ith layer; $J_0(\rho\alpha)$ is the zero-order Bessel function of the first kind. The coefficients in (3), A, B, C_k, D_k, are

the indefinite functions depending on the load, the parameter α and the load time. To determine these functions, the boundary conditions as follows are used:
on the surface, for $z = 0$

$$\sigma_z = \begin{cases} -P(r), & R \ge r, \\ 0, & R < r; \end{cases} \quad (5)$$

on the boundary between the layers $i - 1$ and i:

$$\sigma_z^i = \sigma_z^{i-1}, \; w_i = w_{i-1}, \; u_i = u_{i-1}; \quad (6)$$

in infinity, for $z \to \infty$

$$\sigma_z \to 0, \; w \to 0, \; u \to 0. \quad (7)$$

The set of functions (4) meets the biharmonic equation in cylindrical coordinates (3); therefore, the equations as follows can be used to calculate the movement tensor components:

$$w_i = -\frac{1+\mu_i}{E_i h^2}\int_0^\infty \{ A + B(2+2\mu_i - \eta) + \sum_{k=2} \{C_k[(1- 2\mu_i)(1-e^{-2\lambda_k}) - \lambda_k(1+e^{-2\lambda_k}) - D_k[2(1-2\mu_i)\times (1+e^{-2\lambda_k}) - \lambda_k(1-e^{-2\lambda_k})]]\}\}\alpha^2 e^{-\alpha\eta} J_0(\rho\alpha)d\alpha - \frac{(1+\mu_i)(E_i - H_i)}{E_i^2 h^2 n_i}\int_o^t\int_0^\infty \{A + B(2+2\mu_i - \eta) + \sum_{k=2}\{C_k[(1-2\mu_i)(1-e^{-2\lambda_k}) - \lambda_k(1+e^{-2\lambda_k})] - D_k[2(1-\mu_i)(1+e^{-2\lambda_k}) - \lambda_k(1-e^{-2\lambda_k}))]\}\}\times \alpha^2 e^{\frac{-H_i(t-\tau)}{E_i n_i}-\alpha\eta} J_0(\rho\alpha)\,d\alpha\,d\tau; \quad (8)$$

for horizontal movements:

$$u_i = -\frac{1+\mu_i}{E_i h^2}\int_0^\infty \{ A - B(1-2\mu_i + \eta) + \sum_{k=2}\{C_k[2(1- \mu_i)(1+e^{-2\lambda_k}) + \lambda_k(1-e^{-2\lambda_k})] + D_k[(2\mu_i - 1)\times (1-e^{-2\lambda_k})] - \lambda_k(1+e^{-2\lambda_k})]]\}\}\alpha^2 e^{-\alpha\eta} J_1(\rho\alpha)d\alpha - \frac{(1+\mu_i)(E_i - H_i)}{E_i^2 h^2 n_i}\int_0^t\int_0^\infty \{A - B(1-2\mu_i + \eta) + \sum_{k=2}\{C_k[2(1-\mu_i)(1+e^{-2\lambda_k}) + \lambda_k(1-e^{-2\lambda_k})] + D_k[(2\mu_i - 1)(1-e^{-2\lambda_k}) - \lambda_k(1+e^{-2\lambda_k})]\}\}\times \alpha^2 e^{-\frac{\alpha\eta + H_i(t-\tau)}{E_i n_i}} J_1(\rho\alpha)\,d\alpha\,d\tau, \quad (9)$$

where $\beta = R / h$; $J_1(\rho\alpha)$ is the first-order Bessel function of the first kind. As it was demonstrated above, the boundary conditions are used to determine the functions $A(\alpha,t)$, $B(\alpha,t)$, $C_k(\alpha,t)$, $D_k(\alpha,t)$.

The number of these equations is $2(2n - 1)$; however, with the notation used in this solution for the stress function $\phi_i(r,z)$, the boundary conditions in infinity and the continuity conditions for the stress σ_z are identically true.

The integral transformations result in analytical equations that are applicable to calculate the movement tensor components in any point of the viscoelastic layered half-space, taking the creep of used materials into consideration. These equations have been generalized for the uniform, two-layer and three-layer viscoelastic half-spaces. The boundary conditions (5), (6) were used to obtain the sets of linear equations with variable coefficients that can be used to calculate the values of functions, $A(\alpha, t)$, $B(\alpha, t)$, $C_k(\alpha, t)$, $D_k(\alpha, t)$.

For the fixed parameters α and t, the solution is proved to be unique.

Let's consider the case when the uniform viscoelastic half-space with the parameters E, H, n and μ is exposed to the time-constant vertical load uniformly distributed within the circle of radius R; that can be conveniently described by the Fourier-Bessel integral:

$$P(r) = P_0\beta\int_0^\infty J_1(\beta\alpha)J_0(\rho\alpha)\,d\alpha. \quad (10)$$

If assumed $i = 1$ in (8) and (9), this problem of determining all components of the movement tensor in any point of the isotropic viscoelastic half-space is a special case of the problem considered above. Omitting the intermediate calculations, the final results are as follows:
vertical movements:

$$w = -\frac{1+\mu}{E}\int_0^\infty \{ A + B[2(1-\mu) + \alpha(z-1)]\}e^{-\alpha z}\alpha^2 J_0(\alpha r)d\alpha - \frac{(1+\mu)(E-H)}{E^2 n}\times \int_o^t\int_0^\infty \{A + B[2(1-\mu) + \alpha(z-1)]\}\alpha^2 \times e^{\frac{-H(t-\tau)}{En}-\alpha z} J_0(\alpha r)\,d\alpha\,d\tau; \quad (11)$$

horizontal movements:

$$u = -\frac{1+\mu}{E}\int_0^\infty \{ A - B[(1-2\mu) - \alpha(z-1)]\}e^{-\alpha z}\alpha^2 J_1(\alpha r)d\alpha + \frac{(1+\mu)(H-E)}{E^2 n}\times \int_o^t\int_0^\infty \{A - B[2(1-2\mu) - \alpha(z-1)]\}\alpha^2 \times e^{\frac{-H(t-\tau)}{En}-\alpha z} J_1(\alpha r)\,d\alpha\,d\tau. \quad (12)$$

The boundary conditions (5) for the surface ($z = 0$) are used to calculate the indefinite coefficients A and B in (11) and (12).

Resulting from the conditions listed above, the equations for the unknown coefficients are as follows:

$$A = -PR\alpha^{-2} J_1(R\alpha);$$
$$B = -PR\alpha^{-3} J_1(R\alpha). \quad (13)$$

Substituting the determined A and B values to (11) and (12), we have the formulae applicable to calculate the movement tensor components in any point of the isotropic viscoelastic half-space:
vertical movements:

$$w = -\frac{PR(1+\mu)}{H}(1 + \frac{H-E}{E} e^{-\frac{Ht}{En}} \times$$
$$\times \int_0^\infty (z + \frac{2(1-\mu)}{\alpha}) e^{-\alpha z} J_0(\alpha r) J_1(\alpha R) d\alpha; \quad (14)$$

horizontal movements:

$$w = -\frac{PR(1+\mu)}{H}(1 + \frac{H-E}{E} e^{-\frac{Ht}{En}} \times$$
$$\times \int_0^\infty (z + \frac{2(1-\mu)}{\alpha}) e^{-\alpha z} J_0(\alpha r) J_1(\alpha R) d\alpha. \quad (15)$$

The differential calculus procedures are used to calculate the maximum for the function $w(r)$. For this purpose, the first derivative of $w(r)$ with respect to r shall be set equal to zero. For each α value, the vertical movement along z axis is maximum when $r = 0$, i.e. in the center of a circle of any radius R. Under this condition, the equation (11) for the subsidence in the circle center is simplified because

$$\int_0^\infty (z + \frac{2(1-\mu)}{\alpha}) e^{-\alpha z} J_0(\alpha r) J_1(\alpha R) d\alpha = 2(1-\mu),$$

resulting in the formula applicable to calculate the subsistence under the center of the circle exposed to the time-constant vertical continuous load P:

$$w(t) = \frac{PR(1-\mu^2)}{H}(1 + \frac{H-E}{E} e^{-\frac{Ht}{En}}). \quad (16)$$

Conclusion. These solutions have been generalized for uniform, two-layer and three-layer viscoelastic half-spaces. For these cases, the algorithms and the software have been designed for calculation of the numerical values of movement tensor components.

These algorithms and the software suite have been accepted by the State Fund of Algorithms and Software (Registration No. RG P0015) for application in the engineering design organizations.

The proposed procedures for determination of pavement and roadbed movement tensor components can be applied by the road design organizations within the scope of pavement design methods based on two limit states. These procedures, in combination with the existing design methods, make it possible to consider the real properties of materials more completely and to avoid inacceptable creep deformations throughout the pavement‘s service life.

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