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CALCULATION OF CHARACTERISTICS OF TURBULENT POISEUILLE FLOW

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A previously described approach is used to calculate decay of a laminar flow into individual turbulized liquid layers. The minimum turbulence scale, turbulent viscosity, and frequency spectrum are determined.

In a previous study [1] the present author proposed a new approach to description of the transition of laminar flow of an incompressible liquid into turbulent flow. That approach is based upon introduction of a distribution function $f(\mathbf{r}'|\mathbf{v}(\mathbf{r}))$, which has the sense of the density of the probability that near the point $\mathbf{r}'$, the liquid will have a transport velocity $\mathbf{v}$ corresponding to the solution of the Navier-Stokes equation for the point $\mathbf{r}$. Thus, the original well-defined hydrodynamic description is complemented by probability relationships which reflect the existence of an intrinsic liquid fluctuation mechanism.

Taking a Gaussian law for the function f, it can be established that the dispersion characteristics are determined by the viscous stress tensor. As is well known (see [2]), the latter defines the production of entropy due to internal dissipative processes. Below we will consider a steady state flow of isothermal incompressible isotropic liquid. In this case for the characteristics referred we have

$$\rho = \frac{2aT}{\eta}\sigma,\ a = \frac{2\beta\eta^2}{R\left(\frac{dp}{dx}\right)^2_{cr}},\ \beta = \text{const}. \tag{1}$$

Turbulization of the laminar Poiseulle flow develops upon satisfaction of two conditions for two adjacent coaxial liquid layers:

$$\rho_1(y_1) - \rho_2(y_2) > \beta(y_2 - y_1), \tag{2}$$

$$\Phi_1(y_1^*,\ \rho_1) = \Phi_2(y_2^*,\ \rho_2),\ y_1^* = y_2^* \in b_2. \tag{3}$$

Here the y-axis is directed from the inner surface of the tube along a radius, the coordinate y^* is determined by the point of intersection of two integral distribution curves on the segment b_2, equal to the thickness of the second layer (the first layer is adjacent to the tube surface ($y_1 < y_2$) and correspondingly $\rho_1 > \rho_2$).

For qualitative estimates condition (3) can be reduced to the simpler expression

$$\rho_1 - \rho_2 \geqslant \frac{\rho_1(b_1 + b_2)}{b_1 + 2b_2}. \tag{4}$$

If we represent the characteristic velocity of a hypothetical laminar flow at a given pressure gradient (head) at one of the points in the layer b_k (which is defined by the condition of conservation of flow, while k is measured from the wall and takes on the values 1, 2, ..., n) as the sum of the two velocities

$$v_k = \bar{v}_k + \delta v_k, \tag{5}$$

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where $\bar{v}_k$ is the mean velocity of transport motion of the turbulized layer of thickness b_k, then

$$\bar{v}_k = v_k\Phi_k(\gamma) + v_{k-1}(1 - \Phi_k(\gamma)),\ \gamma = \frac{\dfrac{dp}{dx}}{\left(\dfrac{dp}{dx}\right)_{\rm cr}}. \tag{6}$$

Equation (6) can be used to find the mean flow velocity, and then a hydraulic resistance curve with continuous transition from laminar to turbulent flow and characteristic features of the transition region (for the case of decomposition into two layers $b_1 = R/3$ and $b_2 = 2R/3$).

But does the question of how many and what type of layers an initially laminar flow may decay into remain open? With increase in Reynolds number conditions (2) and (3) admit an ever larger number of decomposition variants. This is to be expected.

The search for an answer to this question must without a doubt be related to a variation principle. The Prigozhin principle of minimum entropy production in the steady state [3] might serve as such. However direct calculation of the viscous stress tensor required for this purpose is difficult to perform, if at all possible for the viscous flow. To a certain degree this difficulty can be circumvented if we limit ourselves to study of the difference in mean values of entropy production of the hypothetical laminar and averaged turbulent flows:

$$\bar{\sigma}_L - \bar{\sigma}_r = \frac{\eta}{2T}(\overline{v_{ij}^2} - \overline{\bar{v}_{ij}^2}),\ \bar{\sigma}_L = \frac{R^2\left(\dfrac{dp}{dx}\right)^2}{24\eta T}, \tag{7}$$

where the tensor $\bar{v}_{ij}$ is constructed with the aid of vector (6), while averaging is performed over all possible layers b_k.

Direct calculation of Eq. (7) shows that this difference has a minimum, which is achieved upon spreading of the flow into layers of equal thickness $b_k = R/n$. In fact this minimum is an arbitrary one, since it is related to the requirements of criteria (2), (3). We will now turn directly to this question.

We will consider two adjacent layers at the tube surfade ($b_1 = b_2 = b$). The necessary condition for turbulization, Eq. (2), gives the inequality

$$b < R\left(1 - \frac{1}{\gamma^2}\right),$$

but the sufficient condition (3) shows that b takes on a nonzero minimum value independent of γ (the pressure gradient, and thus, the Reynolds number), i.e., $b_{min} = 0.347R$. Consequently the boundary laminar layer has a thickness of approximately $b/2 = 0.173R$.

The situation is different if we consider two adjacent layers at the cylinder axis ($b_{n-1} = b_n = b$). Here everything occurs as if in reverse: sufficient condition (3) is realized for all values of γ, while it follows from the necessary condition that

$$b_{\rm min} = \frac{R}{\gamma^2}. \tag{8}$$

This will be the minimum Kolmogorov scale for a given pressure bead $\gamma \geqslant \sqrt{2}$ (beyond the limit of the transition region (see [1])). Thus, in the core of the tube fine scale flow turbulization appears, which confirms the well known concepts of [4]. We recall that turbulence is generated, on the contrary, near the tube surface for $\gamma > 1$, where conditions (2), (3) are satisfied first of all.

Finally, we see that decomposition of a laminar flow is related to the known asymmetry. This additional requirement permits use of the difference in entropy production, Eq. (7), to determine the number of turbulized layers n. The latter takes on selected values in accordance with requirement (8) for given values of pressure head. With increase in Reynolds number discontinuous readjustments of the turbulized flow take place.

It now becomes possible to calculate the turbulent viscosity η_T. Following Klimontovich [5], we can write

$$\eta_{\tau} = \eta \left[1 + \frac{\overline{(\delta v_{ij})^2}}{\overline{v}_{ij}^2} \right]. \tag{9}$$

Representing the velocity gradients in analogy to Eq. (5) and replacing them by finite differences

$$\nabla \overline{v}_k = 2 \frac{\overline{v}_k - \overline{v}_{k-1}}{b_{k-1} + b_k} = \frac{2\Delta \overline{v}_k}{b_{k-1} + b_k},$$

we obtain

$$\eta_{\tau} = \eta \frac{\overline{(\Delta v_k)^2}}{(\Delta \overline{v}_k)^2}, \tag{10}$$

where

$$\Delta \overline{v}_k = v_k \Phi_k - v_{k-1}(1 - \Phi_k - \Phi_{k-1}) + v_{k-2}(1 - \Phi_{k-1}). \tag{11}$$

With the aid of these three integral distribution functions we calculate the mean value of the square of the difference in velocities in two adjacent layers $(v_k - \overline{v}_k)^2$.

The original definition of the critical Reynolds number $Re_{cr}{}^0$, corresponding to the minimum turbulence scale b_{min}, transforms to

$$Re^0_{cr} = \frac{\overline{v}_n b_{min}}{\nu_{\tau}} = \frac{\overline{v}_n R (\Delta \overline{v}_n)^2}{\nu_{\tau} \gamma^2 \overline{(\Delta v_n)^2}}. \tag{12}$$

As an example we will consider decomposition of a flow into three layers (at $\gamma = \sqrt{3}$ beyond the maximum of the hydraulic resistance curve $b_{min} = R/3$ and $b_1 = b_2 = b_3 = b_{min}$). We define the velocities for the midpoints of the intervals

$$v_1 = \frac{\left|\frac{dp}{dx}\right| R^2}{4\eta}\left(1 - \frac{25}{4\gamma^4}\right), \quad v_2 = \frac{\left|\frac{dp}{dx}\right| R^2}{4\eta}\left(1 - \frac{9}{4\gamma^4}\right),$$

$$v_3 = \frac{\left|\frac{dp}{dx}\right| R^2}{4\eta}\left(1 - \frac{1}{4\gamma^4}\right).$$

Then

$$\frac{\overline{(\Delta \overline{v}_3)^2}}{(\Delta \overline{v}_3)^2} = \frac{64(\Phi_3 - 2\Phi_2 + 2)}{A(\gamma)\Phi_2 + B(\gamma)\Phi_3 + C(\gamma)\Phi_2\Phi_3}, \tag{13}$$

where A, B, C are eighth degree polynomials in γ. Hence it follows that $Re^0_{cr} \ll Re$ even for $\gamma \geq \sqrt{3}$ (the difference being greater than two orders of magnitude).

The theory developed here is based on a multifrequency turbulence mechanism. Two frequency spectra can be traced: one related to velocity pulsations, the other, to the alternation process. The character of the frequencies depends on the concrete realization of the possible laminar flow decomposition. In each individual case of decomposition there will be as many frequencies as the number of turbulized layers formed. Each of these will have two frequencies, although from some values of γ the alternation frequencies vanish.

If the initially laminar flow decomposes into two layers ($R = b_1 + b_2$), then in the transitional region the alternation process is characterized by a single frequency $\omega^{\binom{2}{1}}(\gamma)$. The value of this frequency is determined by the relative lifetimes of the laminar and turbulent flows in the layer b_2, or the probabilities $\Phi(y^*, \gamma)$ and $1 - \phi(\gamma)$ [1]. The frequency

$\omega_1^{(2)} = 0$ at $\phi = 1$ (where $\gamma = 1$ is the commencement of turbulization) and at $\phi = 0$ ($\gamma = \sqrt{2}$ is the second critical point). On the other hand, the maximum value is achieved at $\phi = 1/2$. This then permits us to approximate the function $\omega(\gamma)$ in the form

$$\omega = \omega_{max} \sin \frac{2\pi\Delta t}{T}, \tag{14}$$

where

$$\frac{2\Delta t}{T} = 1 - \Phi(y^*, \gamma).$$

When the flow decomposes into three layers (with increase in γ) two alternation frequencies $\omega_1^{(3)}$ and $\omega_2^{(3)}$ develop (the first layer from the tube surface interacts with the second (inner), and the second with the third). These frequencies are defined in terms of the functions $\phi_2(\gamma)$ and $\phi_3(\gamma)$. Thus.

$$\omega_1^{(3)} = \omega_{1max}^{(3)} \sin \pi (1 - \Phi_2). \tag{15}$$

With increase in γ (pressure head) the frequency $\omega_1^{(3)}$ vanishes first.

The velocity pulsation frequency spectrum is defined by that portion of the laminar flow kinetic energy which is related to retardation of translational liquid transport and thus, to formation of vortex motion. This portion of the kinetic energy is defined for each individual turbulized layer by the now known difference in the squares of the velocities $v_k^2 - \bar{v}_k^2$. Therefore the pulsation frequency

$$\omega \sim \frac{(v_k^2 - \bar{v}_k^2)^{\frac{1}{2}}}{b_k}. \tag{16}$$

Then in the laminar boundary layer $\omega \sim \gamma$ (b is independent of γ), while in the tube core $\omega \sim \gamma^3$, since both the velocity and the scale b depend on the pressure drop γ in accordance with Eq. (8).

NOTATION

x, y, coordinates; r radius-vector; v velocity: v_{ij}, velocity derivative tensor; R, tube radius; f, differential distribution function; ϕ distribution function integral; ρ mean square deviation; σ, entropy production; T, temperature; η, shear viscosity coefficient; ν, kinematic viscosity coefficient; p, pressure; γ, ratio of Reynolds number Re to critical value Re_{cr}, equal to the ratio of the corresponding pressure heads (gradients); b, turbulent layer thickness; ω, frequency.

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